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SOME THREE-DIMENSIONAL INCLUSION PROBLEMS IN ELASTICITY

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M. K. Kassir[2] and G. C. Sih[3]

Abstract

The theory of potential functions is applied to solve a number of three-dimensional problems involving sheet-like inclusions embedded in elastic solids. Two types of inclusions are considered; namely, that of a rigid elliptical disk and a rigid sheet containing an elliptical hole. By varying the ellipticity of the disk and hole, certain information on the general character of the stresses around a plane inclusion of arbitrary shape may be obtained. More precisely, if reference is made to a suitable coordinate system, the functional forms of the stresses in the close neighborhood of the inclusion border can be expressed independently of uncertainties of both the inclusion geometry and of the applied stresses or displacements. In general, the intensification of the local stresses can be described by three parameters which may be used to establish criteria for the failure of the solid containing the inclusions.

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Introduction

During the past few decades, considerable attention has been devoted to the solution of two- and three-dimensional problems of stress concentrations around inclusions of a variety of shapes. Since the literature on this subject is exhaustive, only those works which are pertinent to the present study will be cited.

The problem of a thin rigid circular disk embedded in an infinite solid and subjected to a constant displacement normal to its plane was solved by Collins [1]. His results are equivalent to the slow steady motion of a rigid disk in a viscous fluid. In a recent paper, Keer [2] has considered a similar problem in which the disk is displaced in its own plane. The case of an infinite solid containing a rigid sheet with a circular hole was also discussed in [2]. The disturbance of an ellipsoidal inclusion in an otherwise uniform stress field was examined by Eshelby [3,4]. In the limit as one of the principal axes of the ellipsoid vanishes, the solution to the problem of a flat elliptical disk may be deduced from the work in [3,4].

For the purpose of assessing the strength degradation of solids due to the presence of disk-shaped inclusions, it is important to have a knowledge of the singular behavior of the stresses near the sharp edges of the inclusions. To this

end, the present investigation is concerned primarily with the determination of stress solutions of the following boundary-value problems:

(1) A plane inclusion of elliptical shape in an otherwise uniform tensile field.

(2) Elliptical disk displaced in its own plane.

(3) Displacement given to a rigid sheet with an elliptical hole.

(4) Elliptically-shaped disk displaced out of its own plane.

Referring to a system of Cartesian coordinates x,y,z, the z-axis will be directed normal to the plane of discontinuity which is bounded by the ellipse

$$x^2/a^2 + y^2/b^2 = 1, \quad z = 0 \tag{1}$$

where a and b are the major and minor semi-axes of the ellipse, respectively. The center of the ellipse is located at the origin of the coordinate system. The rectangular components of displacement u_x, u_y, u_z and stress σ_{xx}, σ_{yy}, ---, τ_{zx} are assumed to be continuously differentiable at all interior points of the solid and take definite values on either side of the ellipse except that on the periphery of the ellipse the stresses may become infinitely large. At

large distances from the origin, all the stresses and displacements tend to zero. The problem is to find a suitable solution of the Navier's equation of linear elasticity for a homogeneous, isotropic body.

In the absence of body forces, the displacement vector $\underset{\sim}{u}$ is governed by the equation

$$\nabla^2 \underset{\sim}{u} + \frac{1}{1-2\nu} \nabla\nabla . \underset{\sim}{u} = 0 \qquad (2)$$

where ν is Poisson's ratio. The gradient and Laplacian operators in three-dimensions are denoted by ∇ and ∇^2, respectively. For problems exhibiting symmetry about the xy-plane, which contains the surface of discontinuity, the displacement vector $\underset{\sim}{u}$ may be expressed in terms of a vector potential $\underset{\sim}{\phi}$ with components ϕ_x, ϕ_y, ϕ_z and a scalar potential ψ [5]:

$$\underset{\sim}{u} = \underset{\sim}{\phi} + z\nabla\psi \qquad (3)$$

Hence, it is not difficult to verify that eq. (2) can be satisfied by taking

$$\frac{\partial\psi}{\partial z} = - \frac{1}{3-4\nu} \nabla \cdot \underset{\sim}{\phi} \qquad (4)$$

and

$$\nabla^2 \phi = 0, \ \nabla^2 \psi = 0$$

The displacement vectors for problems possessing symmetry with respect to the yz- and zx- planes may be obtained from eqs. (3) and (4) by cyclic permutation of the variables x,y,z. For instance, the representation

$$\underset{\sim}{u} = \underset{\sim}{\phi}' + x\nabla\psi' \ , \ \frac{\partial \psi'}{\partial x} = - \frac{1}{3-4\nu} \nabla \cdot \underset{\sim}{\phi}' \tag{5}$$

applies to problems with symmetry about the yz-plane. In eq. (5), $\underset{\sim}{\phi}'$ and $\underset{\sim}{\psi}'$ satisfy the Laplace equation in three-dimensions.

It should be mentioned that eq. (3) or eq. (5) is a special representation of the more general solution of Papkovitch [6]:

$$\underset{\sim}{u} = 4(1-\nu)\, \underset{\sim}{B} - \nabla\, (\underset{\sim}{R} \cdot \underset{\sim}{B} + B_o) \tag{6}$$

where $\underset{\sim}{R}$ is the position vector. Denoting the components of $\underset{\sim}{B}$ by B_x, B_y, B_z, the Papkovitch functions are related to $\underset{\sim}{\phi}$ and ψ in eq. (3) as

$$\phi_x = - \frac{\partial B_o}{\partial x} \ , \ \phi_y = - \frac{\partial B_o}{\partial y} \ , \ \phi_z = - \frac{\partial B_o}{\partial z} + (3-4\nu) B_z ,$$

$$\psi = B_z$$

and the two components B_x, B_y are taken to be zero.

Once the displacements are known, the stress tensor $\underset{\sim}{\sigma}$ follows directly from the stress-displacement relation

$$\underset{\sim}{\sigma} = \mu \left[\frac{2\nu}{1-2\nu} (\nabla \cdot \underset{\sim}{u})\, \underset{\sim}{I} + \nabla \underset{\sim}{u} + \underset{\sim}{u} \nabla\right] \tag{7}$$

in which μ is the shear modulus of the material and $\underset{\sim}{I}$ is the isotropic tensor.

Triaxial Tension Of Elliptical Disk

Consider an infinite solid with an elliptical disk lying in the xy-plane. The z-axis pierces through the center of the disk whose surfaces are subjected to the displacements

$$Eu_x = -[\sigma_1-\nu(\sigma_2+\sigma_3)]x,$$

$$Eu_y = -[\sigma_2-\nu(\sigma_3+\sigma_1)]y,\ Eu_z = 0 \tag{8}$$

for

$$z = 0 \text{ and } x^2/a^2 + y^2/b^2 \leq 1$$

The Young's modulus is denoted by E. Now, the negative of the displacements in eq. (8) correspond precisely to those of a uniform state of stress in a solid with the disk absent, i.e.,

$$\sigma_{xx} = \sigma_1,\ \sigma_{yy} = \sigma_2,\ \sigma_{zz} = \sigma_3,\ \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \tag{9}$$

Superposition of the solutions of the two preceeding problems will leave both faces of the disk free from displacement and will yield the result to the problem of a thin rigid elliptical disk in an otherwise uniform state of stress. Hence, it suffices to solve the non-trivial second fundamental problem owing to the boundary conditions given by eq. (8).

Let $f(x,y,z)$ be a harmonic function such that

$$\phi_x = (3-4\nu)\frac{\partial f}{\partial x}, \quad \phi_y = (3-4\nu)\frac{\partial f}{\partial y}, \quad \phi_z = 0, \quad \psi = \frac{\partial f}{\partial z} \tag{10}$$

From eq. (3), the displacements become

$$u_x = \frac{\partial F}{\partial x}, \quad u_y = \frac{\partial F}{\partial y}, \quad u_z = z\frac{\partial^2 f}{\partial z^2} \tag{11}$$

in which F is defined as

$$F = (3-4\nu) f + z\frac{\partial f}{\partial z}$$

Upon substitution of eq. (11) into (7) gives the stress components

$$\frac{\sigma_{xx}}{2\mu} = -2\nu\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 F}{\partial x^2}, \quad \frac{\sigma_{yy}}{2\mu} = -2\nu\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 F}{\partial y^2},$$

$$\frac{\sigma_{zz}}{2\mu} = -2(2-\nu)\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 F}{\partial z^2}, \quad \frac{\tau_{xy}}{2\mu} = \frac{\partial^2 F}{\partial x \partial y},$$

$$\frac{\tau_{yz}}{2\mu} = -2(1-\nu)\frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 F}{\partial y \partial z},$$

$$\frac{\tau_{xz}}{2\mu} = -2(1-\nu)\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 F}{\partial x \partial z} \tag{12}$$

To determine the only unknown function $f(x,y,z)$, ellipsoidal coordinates ξ, η, ζ will be employed. The rectangular coordinates x,y,z of any point will be expressed in terms of the triply orthogonal system ξ, η, ζ in the form [7]

$$a^2(a^2-b^2)x^2 = (a^2+\xi)(a^2+\eta)(a^2+\zeta)$$

$$b^2(b^2-a^2)y^2 = (b^2+\xi)(b^2+\eta)(b^2+\zeta)$$

$$a^2b^2z^2 = \xi\eta\zeta \tag{13}$$

where

$$\infty > \xi \geq 0 \geq \eta \geq -b^2 \geq \zeta \geq -a^2$$

In the plane $z = 0$, the inside of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is given by $\xi = 0$, and the outside by $\eta = 0$.

Making use of eqs. (11) and (13), the boundary conditions, eq. (8), become

$$(3-4\nu)\,\frac{\partial f}{\partial x} = -\,\frac{1}{E}\,[\sigma_1-\nu(\sigma_2+\sigma_3)]x, \quad \xi = 0$$

$$(14)$$

$$(3-4\nu)\,\frac{\partial f}{\partial y} = -\,\frac{1}{E}\,[\sigma_2-\nu(\sigma_3+\sigma_1)]y, \quad \xi = 0$$

which implies that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\,\frac{\partial^2 f}{\partial z^2} = \text{constant}, \quad \xi = 0$$

The solution of this problem can be obtained from the known result for the gravitational potential at an external point of a uniform elliptical plate [8], i.e.,

$$f(x,y,z) = \frac{A_1}{2}\int_{\xi}^{\infty}\left[\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1\right]\frac{ds}{\sqrt{Q(s)}} \tag{15}$$

where

$$Q(s) = s(a^2+s)(b^2+s)$$

For subsequent use, the following partial derivatives are computed:

$$\frac{\partial f}{\partial x} = \frac{2A_1}{a^3k^2} [u - E(u)]x$$

$$\frac{\partial f}{\partial y} = \frac{2A_1}{a^3k^2k'^2} [E(u) - k'^2u - k^2 \cdot \frac{\text{snu cnu}}{\text{dn u}}] y \tag{16}$$

The variable u is related to the ellipsoidal coordinate ξ by

$$\xi = a^2(\text{sn}^{-2} u-1)$$

and

$$E(u) = \int_0^u \text{dn}^2t \; dt$$

The quantities snu, cnu, ---, represent the Jacobian elliptic functions and k, k' stand for

$$ak = (a^2-b^2)^{1/2}, \; ak' = b$$

A glance at eqs. (14) and (16) shows that the constants A_1 in eq. (15) cannot be evaluated uniquely. For this reason, the additional solution

$$u_x = - A_2x, \; u_y = A_2y, \; u_z = 0 \tag{17}$$

will be introduced. The sum of eqs. (14) and (17) renders a system of two algebraic equations for the two unknown constants A_1 and A_2 which yields

$$A_1 = -\frac{ab^2}{E(k)} \cdot \frac{(1-\nu)(\sigma_1+\sigma_2) - 2\nu\sigma_3}{4\mu\,(1+\nu)(3-4\nu)}$$

$$A_2 = \frac{\sigma_1-\sigma_2}{4\mu} - \frac{3-4\nu}{a^3k^2}\left[\left(1+\frac{a^2}{b^2}\right)E(k) - 2K(k)\right] A_1 \tag{18}$$

where K(k) and E(k) are the complete elliptical integrals of the first and second kind associated with the modulus k, respectively.

When the stress state

$$\sigma_{xx} = -2\mu A_2,\ \sigma_{yy} = 2\mu A_2,\ \sigma_{zz} = \tau_{xy} = --- = 0$$

is added onto eqs. (12), the contact stresses for $\xi = 0$ may be calculated[4]. The normal stresses

$$\begin{bmatrix} (\sigma_{xx})_{\xi=0} \\ (\sigma_{yy})_{\xi=0} \end{bmatrix} = [\pm]\left(\frac{\sigma_2-\sigma_1}{2}\right) - \frac{3}{2}\left[\frac{(1-\nu)(\sigma_1+\sigma_2) - 2\nu\sigma_3}{(1+\nu)(3-4\nu)}\right]$$

$$(\sigma_{zz})_{\xi=0} = (1-2\nu)\,.\left[\frac{(1-\nu)(\sigma_1+\sigma_2) - 2\nu\sigma_3}{(1+\nu)(3-4\nu)}\right] \tag{19}$$

---

[4]The higher order derivatives of the function f(x,y,z) can be found in a paper by Kassir and Sih [9].

are found to be independent of the geometry of the elliptical disk. For $\eta = 0$, i.e., outside of the ellipse $x^2/a^2 + y^2/b^2 = 1$, σ_{xx}, σ_{yy}, and σ_{zz} become singular on the edge of the disk. Further, the stress exerted by the surrounding material on the disk in the z-direction vanishes if the material is incompressible. The shear stresses on the disk are given by

$$(\tau_{xy})_{\xi=0} = 0$$

$$(\tau_{xz})_{\xi=0} = 2(1-\nu)b\left[\frac{(1-\nu)(\sigma_1+\sigma_2) - 2\nu\sigma_3}{(1+\nu)(3-4\nu)E(k)}\right][x^2/a^2].(1-x^2/a^2-y^2/b^2)^{-\frac{1}{2}}$$

$$(\tau_{yz})_{\xi=0} = 2(1-\nu)b\left[\frac{(1-\nu)(\sigma_1+\sigma_2) - 2\nu\sigma_3}{(1+\nu)(3-4\nu)E(k)}\right][y^2/b^2].(1-x^2/a^2-y^2/b^2)^{-\frac{1}{2}} \tag{20}$$

While both τ_{xz}, τ_{yz} are zero for $\eta = 0$, they are unbounded on the boundary of the disk for $\xi = 0$ as shown in eq. (20).

In the limiting case of $a = b$, $E = K = \pi/2$, the constants A_1 and A_2 in eq. (18) take the forms

$$A_1 = -\frac{a^3}{2\pi\mu}\cdot\frac{(1-\nu)(\sigma_1+\sigma_2) - 2\nu\sigma_3}{(1+\nu)(3-4\nu)}, \quad A_2 = -\frac{\sigma_2-\nu(\sigma_1+\sigma_3)}{2\mu(1+\nu)}$$

and eqs. (19) reduce to the results for a penny-shaped disk given by Collins [1]. The shear stresses in eq. (20) may be combined to yield

$$\sigma_{rz} = \pm 4(1-\nu) \cdot \left[\frac{(1-\nu)(\sigma_1+\sigma_2) - 2\nu\sigma_3}{(1-\nu)(3-4\nu)\pi}\right] \frac{r/a}{\sqrt{1-(r/a)^2}} ,$$

$$0 < r < a, \ z = 0$$

where $\sigma_{rz} = 0$ for $r > a$, $z = 0$. The plus and minus signs refer to the upper and lower faces of the disk, respectively.

Returning to the problem of finding the stress distribution in an infinite solid containing a thin rigid disk under triaxial tension at infinity, it is necessary to express the constants A_1 and A_2, explicitly, in terms of the applied stresses at infinity

$$\sigma_{xx} = \sigma_1^{\infty}, \ \sigma_{yy} = \sigma_2^{\infty}, \ \sigma_{zz} = \sigma_3^{\infty}$$

which are related to σ_1, σ_2, σ_3 in eq. (18) as

$$\sigma_1^{\infty} = \sigma_1 - 2\mu A_2, \ \sigma_2^{\infty} = \sigma_2 + 2\mu A_2, \ \sigma_3^{\infty} = \sigma_3 \tag{21}$$

Inserting eq. (21) into eq. (18), it can be easily shown that σ_1^{∞}, σ_2^{∞}, σ_3^{∞} cannot be prescribed independently. This restriction can be illustrated by considering two special cases as follows:

Case (i) $\sigma_1 = \sigma_2 = 0$

Let the stresses at infinity be

$$\sigma_{xx} = \sigma_1^{\infty} = -2\mu A_2, \ \sigma_{yy} = \sigma_2^{\infty} = 2\mu A_2, \ \sigma_{zz} = \sigma_3^{\infty}$$

Solving for A_1 and A_2 gives

$$2\mu A_1 = \frac{ab^2\nu}{(1+\nu)(3-4\nu)E(k)}\sigma_3{}^\infty$$
$$2\mu A_2 = -\sigma_1{}^\infty = \sigma_2{}^\infty = -\frac{\nu}{(1+\nu)k^2}\left[2-k^2-2k'^2\frac{K(k)}{E(k)}\right]\sigma_3{}^\infty \tag{22}$$

Case (ii) $\sigma_2 = 0$

Another possible solution can be obtained by specifying

$$\sigma_{xx} = \sigma_1{}^\infty = \sigma_1 - 2\mu A_2,\ \sigma_{yy} = \sigma_2{}^\infty = 2\mu A_2,\ \sigma_{zz} = \sigma_3{}^\infty$$

It follows that

$$2\mu A_1 = \frac{a^3k^2[\nu\sigma_3{}^\infty - (1-\nu)\sigma_1{}^\infty]}{2(3-4\nu)[(1-\nu)K(k) - (1-\nu a^2/b^2)E(k)]}$$
$$2\mu A_2 = \sigma_2{}^\infty = \frac{\nu}{2}(\sigma_1{}^\infty+\sigma_3{}^\infty) - \frac{(1+\nu)}{2}\cdot \frac{[(a^2/b^2)E(k) - K(k)][\nu\sigma_3{}^\infty-(1-\nu)\sigma_1{}^\infty]}{(1-\nu)K(k) - (1-\nu a^2/b^2)E(k)} \tag{23}$$

Eqs. (22) and (23) indicate that the specification of the applied stresses is severely restricted[5]. In the present method

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[5]Such a restriction was also mentioned briefly by Eshelby [4] in his survey article on the problem of the ellipsoidal inclusion.

of analysis of inclusion problems, it appears that only two of the three principal stresses at infinity can be specified independently.

Elliptical Disk Displaced Along Its Major Axis

Let an elliptical disk be embedded in an infinite solid and be placed in the xy-plane. The disk is displaced along its major axis by the amount u_o, a constant. The necessary boundary conditions are

$$
\begin{aligned}
& u_x = u_o \; ; \; u_y = u_z = 0 \; , \quad \xi = 0 \\
& u_z = \tau_{xz} = \tau_{yz} = 0 \; , \qquad \eta = 0
\end{aligned}
\tag{24}
$$

The symmetry conditions suggest the following selection of potential functions:

$$
\phi_x' = -(3-4\nu)g + \frac{\partial h}{\partial x}, \; \phi_y' = \frac{\partial h}{\partial y}, \; \phi_z' = \frac{\partial h}{\partial z}, \; \psi' = g \tag{25}
$$

where ϕ_x', ϕ_y', ϕ_z' are the rectangular components of the vector $\tilde{\phi}'$ in eq. (5). The functions $g(x,y,z)$ and $h(x,y,z)$ satisfy the Laplace equations

$$
\nabla^2 g(x,y,z) = 0 \; , \; \nabla^2 h(x,y,z) = 0
$$

Putting eq. (25) into (5), it is found that

$$
u_x = -4(1-\nu)g + \frac{\partial G}{\partial x} \; , \; u_y = \frac{\partial G}{\partial y} \; , \; u_z = \frac{\partial G}{\partial z} \tag{26}
$$

From eq. (7), the components of stress are obtained:

$$\frac{\sigma_{xx}}{2\mu} = -2(2-\nu)\frac{\partial g}{\partial x} + \frac{\partial^2 G}{\partial x^2}\ , \quad \frac{\sigma_{yy}}{2\mu} = -2\nu\frac{\partial g}{\partial x} + \frac{\partial^2 G}{\partial y^2}\ ,$$

$$\frac{\sigma_{zz}}{2\mu} = -2\nu\frac{\partial g}{\partial x} + \frac{\partial^2 G}{\partial z^2}\ ,$$

$$\frac{\tau_{xy}}{2\mu} = -2(1-\nu)\frac{\partial g}{\partial y} + \frac{\partial^2 G}{\partial x \partial y}\ , \quad \frac{\tau_{yz}}{2\mu} = \frac{\partial^2 G}{\partial y \partial z}\ ,$$

$$\frac{\tau_{zx}}{2\mu} = -2(1-\nu)\frac{\partial g}{\partial z} + \frac{\partial^2 G}{\partial x \partial z} \qquad (27)$$

The appropriate harmonic functions for this problem may be chosen as

$$g(x,y,z) = B_1 \int_\xi^\infty \frac{ds}{\sqrt{Q(s)}} = \frac{2B_1}{a} u\ ,$$

$$h(x,y,z) = B_2 x \int_\xi^\infty \frac{ds}{(a^2+s)\sqrt{Q(s)}} = \frac{2B_2}{a^3 k^2}[u - E(u)]x \qquad (28)$$

Note that $h(x,y,z)$, except for the multiplying constant, represents the derivative of the gravitational potential at an external point of an elliptical disk with respect to x. For the purpose of evaluating the constants B_1 and B_2, the displacement component u_z is computed:

$$u_z = -\frac{2x[\eta\zeta(a^2+\xi)(b^2+\xi)]^{\frac{1}{2}}}{ab(\xi-\eta)(\xi-\zeta)}[B_1 + \frac{B_2}{a^2+\xi}]$$

The condition that u_z vanishes everywhere on the plane $z = 0$ yields

$$B_2 = -a^2 B_1 \tag{29}$$

By virtue of eqs. (24), (26) and (29) for $\xi = 0$, B_1 is found:

$$B_1 = -\frac{u_o}{2} \cdot \frac{ak^2}{[(3-4\nu)k^2+1]K(k) - E(k)} \tag{30}$$

Knowing B_1 and B_2, the displacements and stresses at any point of the solid can be calculated. On the plane $z = 0$, the non-vanishing displacements are

$$(u_x)_{\eta=0} = -\frac{2B_1}{ak^2}\left\{[1+(3-4\nu)k^2]u - E(u)\right.$$

$$\left. + \frac{a(kx)^2}{(\xi-\zeta)(a^2+\xi)}\sqrt{\frac{\xi(b^2+\xi)}{a^2+\xi}}\right\} \tag{31}$$

$$(u_y)_{\eta=0} = -\frac{2B_1 xy}{\xi-\zeta} \cdot \sqrt{\frac{\xi}{(a^2+\xi)(b^2+\xi)}}$$

and the stresses are

$$(\tau_{xz})_{\xi=0} = \frac{8\mu(1-\nu)B_1}{ab}(1-x^2/a^2-y^2/b^2)^{-\frac{1}{2}}$$

$$(\sigma_{zz})_{\eta=0} = -\frac{4\mu(1-2\nu)B_1 x}{\xi-\zeta} \cdot \sqrt{\frac{b^2+\xi}{\xi(a^2+\xi)}} \tag{32}$$

Both τ_{xz} and σ_{zz} are singular on the border of the ellipse $x^2/a^2+y^2/b^2 = 1$, while $\tau_{yz} = 0$ everywhere on the plane $z = 0$.

When $a = b$, $K = E = \pi/2$, eq. (30) simplifies to the form

$$B_1 = - \frac{2au_0}{\pi(7-8\nu)}$$

It can be verified that for $r > a$, $z = 0$, $\xi \to r^2 - a^2$, and $u \to \sin^{-1} (\frac{a}{r})$, eqs. (31) and (32) are in agreement with eqs. (23) and (24) in [2], respectively, except for[6]

$$(\sigma_{zz})_{z=0} = \frac{8\mu(1-2\nu)}{\pi(7-8\nu)} (\frac{u_0}{a}) \cdot \frac{\cos\theta}{(r/a)\sqrt{(r/a)^2 - 1}}, \quad r > a \quad (33)$$

where u_0 corresponds to Δ in [2].

---

[6]Eq. (33) may also be derived directly from eq. (20) in [2] if the order of integration and differentiation is properly observed as follows:

$$(\sigma_{zz})_{z=0} = \frac{1}{2}(1-2\nu)\frac{\partial}{\partial x} [\lim_{z\to 0} \int_{-a}^{+a} \frac{f(t)\, dt}{\sqrt{r^2+(z+it)^2}}], \quad f(t) = - \frac{8\mu u_0}{\pi(7-8\nu)}$$

Carrying out the integration gives

$$(\sigma_{zz})_{z=0} = - \frac{8\mu(1-2\nu)u_0}{\pi(7-8\nu)} \frac{\partial}{\partial x} [\sin^{-1} (\frac{a}{r})]$$

$$= \frac{8\mu(1-2\nu)u_0}{\pi(7-8\nu)} (\frac{a}{r})(r^2-a^2)^{-1/2} \cos\theta$$

Hence, the factor $(1-\nu)$ in eq. (24) of [2] should be replaced by $\cos\theta$.

The foregoing method of solution may also be used to solve the problem of an elliptical disk displaced in an arbitrary direction by a constant amount, say δ_0. If ω denotes the angle between the x-axis and the direction along which the disk is caused to move, then the boundary conditions, eq. (24), may be generalized:

$$u_x = \delta_0 \cos \omega, \; u_y = \delta_0 \sin \omega, \; u_z = 0, \; \xi = 0$$

$$u_z = \tau_{xz} = \tau_{yz} = 0, \; \eta = 0$$

The displacements are expressible in terms of four harmonic functions as

$$u_x = -4(1-\nu)g_1 + \frac{\partial G_o}{\partial x}, \; u_y = -4(1-\nu)g_2 + \frac{\partial G_o}{\partial y}, \; u_z = \frac{\partial G_o}{\partial z}$$

in which

$$G_o = G_1 + G_2, \; G_1 = xg_1 + h_1, \text{ and } G_2 = yg_2 + h_2$$

To satisfy the Laplace equations in three dimensions, $g_j(x,y,z)$ and $h_j(x,y,z)$ are taken in the forms

$$g_j(x,y,z) = C_j \int_{\xi}^{\infty} \frac{ds}{\sqrt{Q(s)}}, \; j = 1, 2$$

$$h_1(x,y,z) = D_1 x \int_{\xi}^{\infty} \frac{ds}{\sqrt{a^2+s}\,\sqrt{Q(s)}},$$

$$h_2(x,y,z) = D_2 y \int_{\xi}^{\infty} \frac{ds}{(b^2+s)\sqrt{Q(s)}}$$

Since the displacement u_z vanishes for $z = 0$, the constants D_j may be expressed in terms of C_j:

$$D_1 = -a^2C_1 \ , \ D_2 = -b^2C_2$$

The remaining unknowns, say C_j ($j = 1, 2$), can be evaluated from the boundary conditions yet to be satisfied and the solution of the problem is essentially complete.

Displacement Of Rigid Sheet With Elliptical Hole

Suppose that two semi-infinite solids are bonded perfectly to a thin rigid sheet with an elliptical opening through which the solids are connected. The sheet is allowed to move in the plane $z = 0$ by a constant amount parallel to the x-axis. The equivalent condition is to specify a constant shear stress $\tau_{zx} = \tau_o$ for $\xi = 0$. For this problem, the following conditions must be satisfied:

$$\begin{aligned} &u_x = u_y = 0,\ \eta = 0;\ u_z = 0,\ z = 0 \\ &\tau_{yz} = 0,\ \tau_{zx} = \tau_o,\ \xi = 0 \end{aligned} \qquad (34)$$

The problem may be formulated in terms of a single function $p(x,y,z)$ which is related to $\underset{\sim}{\phi}$ and ψ in eqs. (3) and (4) as

$$\phi_x = -(3-4\nu)\frac{\partial p}{\partial z},\ \phi_y = \phi_z = 0,\ \psi = \frac{\partial p}{\partial x}$$

where

$$\nabla^2 p(x,y,z) = 0$$

The representation of the components of displacement as given by Trefftz [5] is

$$u_x = -(3-4\nu)\frac{\partial p}{\partial z} + z\,\frac{\partial^2 p}{\partial x^2},\quad u_y = z\,\frac{\partial^2 p}{\partial x \partial y},\quad u_z = z\,\frac{\partial^2 p}{\partial x \partial z} \qquad (35)$$

The stresses corresponding to eq. (35) are given by

$$\frac{\sigma_{xx}}{2\mu} = \frac{\partial}{\partial x}\left[-(3-2\nu)\frac{\partial p}{\partial z} + z\,\frac{\partial^2 p}{\partial x^2}\right],\quad \frac{\sigma_{yy}}{2\mu} = \frac{\partial}{\partial x}\left[-2\nu\frac{\partial p}{\partial z} + z\,\frac{\partial^2 p}{\partial y^2}\right],$$

$$\frac{\sigma_{zz}}{2\mu} = \frac{\partial}{\partial x}\left[(1-2\nu)\frac{\partial p}{\partial z} + z\,\frac{\partial^2 p}{\partial z^2}\right],\quad \frac{\tau_{xy}}{\mu} = \frac{\partial}{\partial y}\left[-(3-4\nu)\frac{\partial p}{\partial z} + 2z\,\frac{\partial^2 p}{\partial x \partial y}\right],$$

$$\frac{\tau_{yz}}{\mu} = \frac{\partial^2}{\partial x \partial y}\left[p + 2z\,\frac{\partial p}{\partial z}\right],\quad \frac{\tau_{zx}}{\mu} = -(3-4\nu)\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2}{\partial x^2}\left[p + 2z\,\frac{\partial p}{\partial z}\right]$$

(36)

On the plane $z = 0$, eq. (34) requires that

$$\frac{\partial p}{\partial z} = 0,\quad \eta = 0$$

$$\frac{\partial^2 p}{\partial x^2} - (3-4\nu)\,\frac{\partial^2 p}{\partial z^2} = \frac{\tau_o}{\mu},\quad \xi = 0 \qquad (37)$$

The first condition in eqs. (37) is satisfied automatically by taking

$$p(x,y,z) = \frac{C}{2}\int_{\xi}^{\infty}\left[\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1\right]\frac{ds}{\sqrt{Q(s)}}$$

while the second condition yields

$$2\mu C = \frac{a^3k^2k'^2\tau_0}{k'^2K(k)+[(3-4\nu)k^2-k'^2]E(k)}$$

Once p(x,y,z) is determined, the displacements and stresses throughout the solid can be computed from eqs. (35) and (36).

For z = 0, both u_y and u_z vanish and

$$(u_x)_{\xi=0} = -\frac{2C\ (3-4\nu)}{ab}(1-x^2/a^2-y^2/b^2)^{1/2},\quad (u_x)_{\eta=0} = 0$$

The stresses on the plane z = 0 are

$$(\sigma_{zz})_{\xi=0} = -\frac{4\mu(1-2\nu)C\ x}{a^3b}(1-x^2/a^2-y^2/b^2)^{-1/2}$$

$$(\tau_{yz})_{\eta=0} = -\frac{2\mu C\ xy}{(\xi-\zeta)\sqrt{Q(\xi)}} \tag{38}$$

$$(\tau_{zx})_{\eta=0} = 2\mu C\left\{-\frac{3-4\nu}{ab^2}\left[\frac{ab^2}{\sqrt{Q(\xi)}} - E(u) + \frac{\mathrm{sn}u\ \mathrm{cn}u}{\mathrm{dn}\ u}\right] + \frac{u-E(u)}{a^3k^2} - \frac{x^2}{(\xi-\zeta)(a^2+\xi)}\sqrt{\frac{b^2+\xi}{\xi(a^2+\xi)}}\right\}$$

and

$$(\sigma_{zz})_{\eta=0} = (\tau_{yz})_{\xi=0} = 0,\quad (\tau_{zx})_{\xi=0} = \tau_0$$

Using L' Hospital's rule, the constant C for a circular hole, a = b, may be recovered:

$$C = \frac{2a^3\tau_0}{\pi\mu(7-8\nu)}$$

Aside from a couple of misprints, $(u_x)_{\xi=0}$, $(\tau_{yz})_{\eta=0}$, and $(\tau_{zx})_{\eta=0}$ check with those given by eqs. (41) and (42) in [2] if τ_0 is identified with σ_0. The expression for

$$(\sigma_{zz})_{z=0} = -\frac{8(1-2\nu)}{\pi(7-8\nu)} \frac{r/a}{\sqrt{1-(r/a)^2}} \tau_0 \cos\theta$$

fails to agree with that of [2] for the same reason as mentioned earlier in footnote (6).

Axial Displacement Of Elliptical Disk

If a thin rigid disk of elliptical shape is given a constant displacement w_0 normal to its plane, then

$$u_x = u_y = 0,\ z = 0;\ u_z = w_0,\ \xi = 0 \tag{39}$$

which suggests that

$$\phi_x = \phi_y = 0,\ \phi_z = -(3-4\nu)q,\ \psi = q \tag{40}$$

Inserting eq. (40) into (3), the result is

$$u_x = z\frac{\partial q}{\partial x},\ u_y = z\frac{\partial q}{\partial y},\ u_z = -(3-4\nu)q + z\frac{\partial q}{\partial z} \tag{41}$$

From eq. (7), it is further found that

$$\frac{\sigma_{xx}}{2\mu} = -2\nu\frac{\partial q}{\partial z} + z\frac{\partial^2 q}{\partial x^2},\ \frac{\sigma_{yy}}{2\mu} = -2\nu\frac{\partial q}{\partial z} + z\frac{\partial^2 q}{\partial y^2},$$

$$\frac{\sigma_{zz}}{2\mu} = -2(1-\nu)\frac{\partial q}{\partial z} + z\frac{\partial^2 q}{\partial z^2},\ \frac{\tau_{xy}}{2\mu} = z\frac{\partial^2 q}{\partial x \partial y}$$

$$\frac{\tau_{yz}}{2\mu} = -(1-2\nu)\frac{\partial q}{\partial y} + z\,\frac{\partial^2 q}{\partial y \partial z},\quad \frac{\tau_{zx}}{2\mu} = -(1-2\nu)\frac{\partial q}{\partial x} + z\,\frac{\partial^2 q}{\partial x \partial z} \qquad (42)$$

The only unknown function $q(x,y,z)$ satisfying

$$\nabla^2 q(x,y,z) = 0$$

can be taken in the form

$$q(x,y,z) = D \int_{\xi}^{\infty} \frac{ds}{\sqrt{Q(s)}} = \frac{2D}{a}\, u \qquad (43)$$

Eqs. (39), (41) and (43) may be combined to give

$$D = -\frac{a w_o}{2(3-4\nu)K(k)}$$

Calculating for the derivatives of $q(x,y,z)$ with respect to x,y,z, i.e.,

$$\frac{\partial \phi}{\partial x} = \frac{a w_o x}{(3-4\nu)(\xi-\eta)(\xi-\zeta)K(k)} \cdot \sqrt{\frac{\xi(b^2+\xi)}{a^2+\xi}}\ ,$$

$$\frac{\partial \phi}{\partial y} = \frac{a w_o y}{(3-4\nu)(\xi-\eta)(\xi-\zeta)K(k)} \cdot \sqrt{\frac{\xi(a^2+\xi)}{b^2+\xi}}\ ,$$

$$\frac{\partial \phi}{\partial z} = \frac{w_o\ (\eta\zeta)^{1/2}}{(3-4\nu)b\ (\xi-\eta)(\xi-\zeta)K(k)} \cdot \sqrt{(a^2+\xi)(b^2+\xi)}$$

and so on ---, the non-trivial displacements and stresses for $z = 0$ are

$$(u_z)_{\xi=0} = w_o,\quad (u_z)_{\eta=0} = \frac{w_o}{K(k)} \cdot [u]_{\eta=0}$$

and

$$(\sigma_{zz})_{z=0^{\pm}} = \mp \frac{4\mu(1-\nu)w_o}{(3-4\nu)b\ K(k)} (1-x^2/a^2-y^2/b^2)^{-1/2}, \quad \xi = 0$$

$$\begin{bmatrix} (\tau_{xz})_{z=0^+} \\ (\tau_{yz})_{z=0^+} \end{bmatrix} = - \frac{2\mu(1-2\nu)w_o}{(3-4\nu)\xi^{1/2}(\xi-\zeta)k\ K(k)} \begin{bmatrix} \sqrt{(a^2+\zeta)(b^2+\xi)} \\ \sqrt{(a^2+\xi)[-(b^2+\zeta)]} \end{bmatrix}, \eta = 0 \tag{44}$$

in which $-(b^2+\zeta)$ is a positive definite quantity. The notations $z=0^+$ and $z=0^-$ refer to the upper and lower faces of the disk, respectively.

The force exerted by the elastic solid to oppose the displacement of the elliptical disk may be found from the integral

$$F_z = \int_{\Sigma}\int [(\sigma_{zz})_{z=0^+} - (\sigma_{zz})_{z=0^-}]\ dxdy \tag{45}$$

The region Σ is bounded by the ellipse $x^2/a^2+y^2/b^2 = 1$. Substituting eq. (44) into (45), F_z is obtained:

$$\begin{aligned} F_z &= - \frac{8\mu(1-\nu)w_o}{(3-4\nu)b\ K(k)} \int_{\Sigma}\int (1-x^2/a^2-y^2/b^2)^{-1/2} dxdy \\ &= - \frac{16\pi\mu(1-\nu)aw_o}{(3-4\nu)K(k)} \end{aligned} \tag{46}$$

In the limit as $a \to b$, eq. (46) reduces to Collin's solution [1] for a circular disk.

Three-Dimensional Stresses Near Inclusion Border

For the purpose of establishing possible failure criteria, the stresses near the border of a plate-like inclusion will be investigated. It is convenient to introduce a rectangular cartesian coordinate system n,t,z such that the origin of this system traverses the periphery of the inclusion. The zn-, nt-, and tz- planes are known, respectively, as the normal, rectifying and osculating planes to the curve which will be taken in the form of an ellipse.

In the immediate vicinity of the inclusion border, the ellipsoidal coordinates ξ, η, ζ can be expressed in terms of the polar coordinates r, θ defined in the nz-plane, where r is the radial distance measured from the edge of the inclusion and θ is the angle between r and the n-axis. The required relationships of ξ, η, ζ to r, θ are[7]

$$\xi = \frac{2abr}{(a^2\sin^2\phi + b^2\cos^2\phi)^{1/2}} \cos^2 \frac{\theta}{2}$$

$$\eta = - \frac{2abr}{(a^2\sin^2\phi + b^2\cos^2\phi)^{1/2}} \sin^2 \frac{\theta}{2} \tag{47}$$

$$\zeta = - (a^2\sin^2\phi + b^2\cos^2\phi)$$

[7]A detailed derivation of eq. (47) is given in [9].

In eq. (47), r is assumed to be small in comparison with a (or b) and ϕ is the angle appearing in the parametric equations of the ellipse, i.e.,

$$x = a \cos \phi, \; y = b \sin \phi$$

Since the derivation of the local stresses is similar to those given by Kassir and Sih [9] for the three-dimensional crack problem, the detail calculations will be omitted here. By means of eq. (47) and the appropriate equations for finding the stresses, the following results are obtained:

$$\sigma_{nn} = + \frac{k_1}{\sqrt{2r}} \cos \frac{\theta}{2} \left(3-2\nu - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right)$$

$$+ \frac{k_2}{\sqrt{2r}} \sin \frac{\theta}{2} \left(2\nu + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) + 0(1)$$

$$\sigma_{zz} = - \frac{k_1}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1-2\nu - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right)$$

$$+ \frac{k_2}{\sqrt{2r}} \sin \frac{\theta}{2} \left(2-2\nu - \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) + 0(1)$$

$$\sigma_{tt} = + \frac{k_1}{\sqrt{2r}} \cdot 2\nu \cos \frac{\theta}{2} + \frac{k_2}{\sqrt{2r}} 2\nu \sin \frac{\theta}{2} + 0(1)$$

$$\tau_{nt} = - \frac{k_3}{\sqrt{2r}} \cos \frac{\theta}{2} + 0(1)$$

$$\tau_{nz} = + \frac{k_1}{\sqrt{2r}} \sin\frac{\theta}{2}\left(2-2\nu + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$+ \frac{k_2}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1-2\nu + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + 0(1)$$

$$\tau_{tz} = \frac{k_3}{\sqrt{2r}} \sin\frac{\theta}{2} + 0(1) \tag{48}$$

Although these stresses were derived from the solution of an elliptically-shaped inclusion, they are in general valid for a plane inclusion of arbitrary shape. Moreover, the inclusion-border stress fields for the four preceding boundary-value problems are included in eq. (48) as special cases.

Now, it is significant to observe that eq. (48) is composed of the linear sum of three distinct stress fields each of which can be associated with a different mode of deformation. Referring to Figs. 1(a) through 1(c), the intensity of the local stresses at the point P caused by the movements of the inclusion in the n-, z-, and t- directions are governed, respectively, by the three parameters k_1, k_2 and k_3. These three modes of displacements are necessary and sufficient to describe all the possible displacements of the inclusion. It will be shown subsequently that the parameters k_j ($j = 1, 2, 3$) depend only upon the prescribed stresses or displacements and the inclusion geometry. The singular behavior of the inclusion-border stresses

is the same as that for a sharp crack. In other words, the $1/\sqrt{r}$ type of stress singularity is preserved. However, unlike the crack problem, the angular distribution of the stresses is a function of the Poisson's ratio of the elastic solid.

A close examination of the stress expressions in eq. (48) reveals that σ_{nn}, σ_{zz}, and τ_{nz} correspond precisely to those obtained by Sih [10][8] for a rigid line inclusion under the conditions of plane strain. In fact, the stress component σ_{tt} is equal to $\nu(\sigma_{nn}+\sigma_{zz})$, a condition which is well known in the analysis of plane strain problems. The shear stresses τ_{nt} and τ_{tz} can be identified with the two-dimensional problem of a line inclusion subjected to longitudinal or out-of-plane shear loads. Hence, the stress state around a plane inclusion in three-dimensions is locally one of plane strain combined with longitudinal shear.

---

[8]The stresses σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ given by eq. (48) in [10] should be transformed into rectangular components σ_{xx}, σ_{yy}, τ_{xy} in accordance with

$$\sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\tau_{xy} = e^{-2i\theta}(\sigma_{\theta\theta}-\sigma_{rr} + 2i\tau_{r\theta})$$

For $\kappa = 3-4\nu$, the functional forms of σ_{xx}, σ_{yy}, τ_{xy} correspond to σ_{nn}, σ_{zz}, τ_{nz} in this paper, respectively.

In general, the three parameters k_j ($j = 1, 2, 3$) will occur simultaneously over the inclusion border. They may be interpretated as a measure of the elevation of stresses due to the presence of thin rigid inclusions embedded in elastic solids. From eq. (48), the formulas

$$k_1 = \frac{1}{1-2\nu} \lim_{r\to 0} \sqrt{2r}\,(\sigma_{zz})_{\theta=0}$$

$$k_2 = \frac{1}{1-2\nu} \lim_{r\to 0} \sqrt{2r}\,(\tau_{nz})_{\theta=0} \tag{49}$$

$$k_3 = \lim_{r\to 0} \sqrt{2r}\,(\tau_{tz})_{\theta=0}$$

are obtained. Eq. (49) may be applied to evaluate k_j for the boundary-value problems solved earlier. Following the work of Kassir and Sih [9], it is found that

(1) Triaxial Tension.

$$k_1 = + \frac{(1-\nu)(\sigma_1+\sigma_2)-2\nu\sigma_3}{(1+\nu)(3-4\nu)E(k)} \left(\frac{b}{a}\right)^{1/2} (a^2\sin^2\phi + b^2\cos^2\phi)^{1/4},$$

$$k_2 = k_3 = 0 \tag{50}$$

(2) Parallel Displacement.

$$k_1 = - \frac{2\mu a k^2 u_o}{[(3-4\nu)k^2+1]K(k)-E(k)} \left(\frac{b}{a}\right)^{1/2} (a^2\sin^2\phi$$

$$+ b^2\cos^2\phi)^{-3/4} \cos\phi,\ k_2 = 0$$

$$k_3 = \frac{4\mu(1-\nu)ak^2u_0}{[(3-4\nu)k^2+1]K(k)-E(k)} \left(\frac{a}{b}\right)^{1/2} (a^2\sin^2\phi$$

$$+ b^2\cos^2\phi)^{-3/4} \sin\phi \tag{51}$$

(3) Rigid Sheet.

$$k_1 = + \frac{2bk^2\tau_0}{[(3-4\nu)k^2-k'^2]E(k)+k'^2K(k)} \left(\frac{b}{a}\right)^{1/2} (a^2\sin^2\phi$$

$$+ b^2\cos^2\phi)^{-1/4} \cos\phi, \; k_2 = 0$$

$$k_3 = \frac{(3-4\nu)ak^2\tau_0}{[(3-4\nu)k^2-k'^2]E(k)+k'^2K(k)} \left(\frac{b}{a}\right)^{1/2} (a^2\sin^2\phi$$

$$+ b^2\cos^2\phi)^{-1/4} \sin\phi \tag{52}$$

(4) Axial Displacement

$$k_1 = 0, \; k_2 = - \frac{2\mu w_0}{(3-4\nu)K(k)} \left(\frac{a}{b}\right)^{1/2} (a^2\sin^2\phi$$

$$+ b^2\cos^2\phi)^{-1/4}, \; k_3 = 0 \tag{53}$$

It is interesting to note that k_j are not constants but functions of position. Eq. (50) is associated with the local displacement shown in Fig. 1(a) while eq. (53) with Fig. 1(b). The displacement modes pertaining to the results in eqs. (51) and (52) are more complicated. For $0 < \phi < \frac{\pi}{2}$, the inclusion

border experiences a combination of the movements illustrated in Figs. 1(a) and 1(c). The parameters k_1 and k_3 attain their maximum values at $\phi = 0$ and $\phi = \frac{\pi}{2}$, respectively.

For problems involving all three parameters k_j ($j = 1, 2, 3$), it is possible to postulate a criterion of failure for rigid inlcusions in the form

$$fcr = f(k_1, k_2, k_3)$$

which states that failure of the material surrounding the inclusion occurs when the combination of k_1, k_2, and k_3 attains some critical value.

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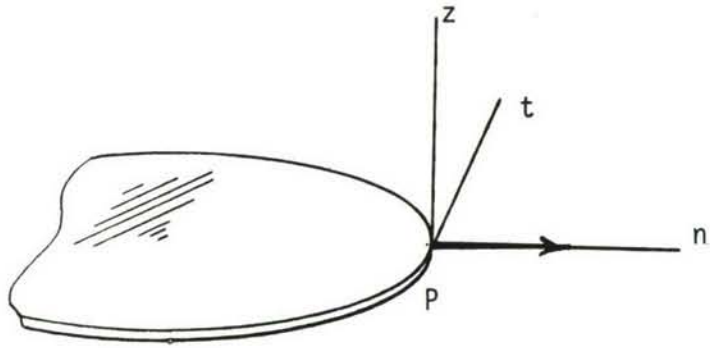
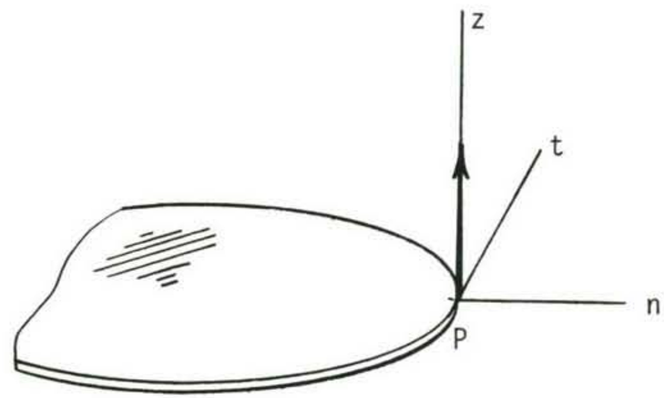
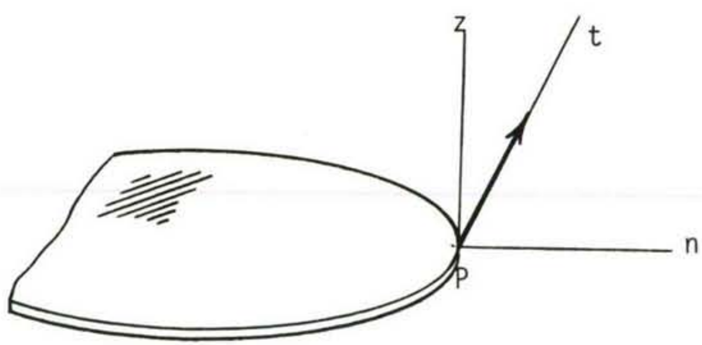


(a) $k_1 \neq 0, \; k_2 = k_3 = 0$.



(b) $k_1 = 0, \; k_2 \neq 0 \; , \; k_3 = 0$.



(c) $k_1 = k_2 = 0, \; k_3 \neq 0$.

Fig. 1 - The Basic Modes of Plane Inclusion Displacements.

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